One Sheet Vector Review

Vectors are numerical objects characterized by a ***magnitude*** and a ***direction***. Vectors can be moved around as long as their length (magnitude) and direction/orientation do not change.

Unit Vectors

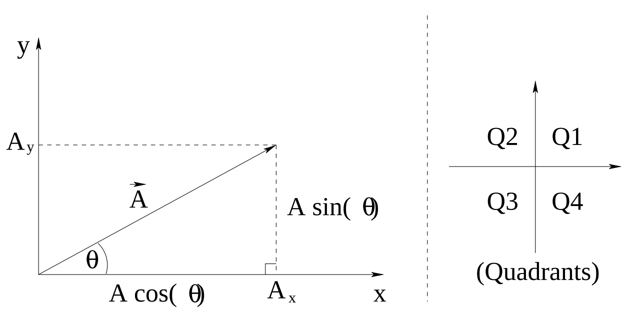
Unit vectors are vectors of length 1 in the fundamental perpendicular directions of your coordinate system. There are two common conventions for representing the unit vectors in the x, y, and z directions:

$$\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}} \quad \text{or} \quad \hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$$

We can stretch these unit vectors to any length we like by multiplying by *scalar* magnitudes, e.g .

$$A_x\hat{\boldsymbol{x}} \quad \text{or} \quad B_y\hat{\boldsymbol{j}} \quad \text{or} \quad C_z\hat{\boldsymbol{z}}$$

2D Vector Representations



Cartesian: $\vec{\boldsymbol{A}} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}}$ or $\vec{\boldsymbol{A}} = (A_x, A_y)$
Polar: $\vec{\boldsymbol{A}} = (A, \theta)$
where:

$$A_x = A\cos(\theta) \qquad A_y = A\sin(\theta)$$

$$A = \sqrt{A_x^2 + A_y^2} \qquad \theta = \tan^{-1}(A_y/A_x)$$

θ is measured positive as the angle between the positive x-axis and the vector in the ***counterclockwise*** direction.

Adding or Subtracting Vectors

Given vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ with components (A_x, A_y) and (B_x, B_y) respectively:

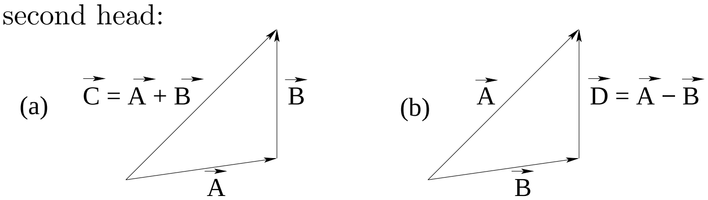
$$\vec{\boldsymbol{C}} = \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = (A_x + B_x)\hat{\boldsymbol{x}} + (A_y + B_y)\hat{\boldsymbol{y}}$$

similarly:

$$\vec{\boldsymbol{D}} = \vec{\boldsymbol{A}} - \vec{\boldsymbol{B}} = (A_x - B_x)\hat{\boldsymbol{x}} + (A_y - B_y)\hat{\boldsymbol{y}}$$

Triangles for Vector Addition

Vectors can graphically be visualized or added by putting the head of one vector at the tail of the other and drawing the *resultant* as the triangle connecting the first tail to the second head:



Useful Trig/Triangles You Should Know

$$\sin(30^\circ) = \cos(60^\circ) = 1/2 \quad \cos(30^\circ) = \sin(60^\circ) = \sqrt{3}/2$$

$$\sin(45^\circ) = \cos(45^\circ) = \sqrt{2}/2$$

($30^\circ = \pi/6$, $45^\circ = \pi/4$)

$$\sin(37^\circ) \approx 0.6 \quad \cos(37^\circ) \approx 0.8$$

(3-4-5 triangle angles are 37°and 53°)

Kinds of Products of (3D) Vectors

Inner or Scalar or Dot Product:

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = A_xB_x + A_yB_y + A_zB_z = AB\cos(\theta)$$

The (scalar) length of a vector is defined to be:

$$A = +\sqrt{A^2} = +\sqrt{\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{A}}} = +\sqrt{A_x^2 + A_y^2 + A_z^2}$$

Cross or Vector Product:

$$|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}| = AB\sin(\theta)$$

and direction from ***right hand rule***, align fingers of right hand with $\vec{\boldsymbol{A}}$, rotate through the smaller angle in the plane into $\vec{\boldsymbol{B}}$, thumb indicates the direction of the cross product, or use the Cartesian representation:

$$\vec{\boldsymbol{C}} = \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = (A_yB_z - A_zB_y)\,\hat{\boldsymbol{x}} + (A_zB_x - A_xB_z)\,\hat{\boldsymbol{y}} + (A_xB_y - A_yB_x)\,\hat{\boldsymbol{z}}$$

To easily remember this last form, note that there are three *cyclic permutations* of the coordinates in alphabetical order: x y z, y z x, z x y. Observe that the ***positive*** term in each parentheses and the unit vector contain x y z in ***cyclic order***: for example $A_xB_y\hat{\boldsymbol{z}}$. The negative terms in each parentheses simply have the first two indices swapped, as in $-A_yB_x\hat{\boldsymbol{z}}$.

One Sheet Calculus Review

Derivatives

$$\frac{du^n}{du} = n\ u^{n-1}$$

$$\frac{de^u}{du} = e^u$$

$$\frac{d\sin(u)}{du} = \cos(u)$$

$$\frac{d\cos(u)}{du} = -\sin(u)$$

$$\frac{d\ln(u)}{du} = \frac{1}{u}$$

Indefinite Integrals

$$\text{for } (n \neq -1) \qquad \int u^n\ du = \frac{u^{n+1}}{n+1}$$

$$\text{for } (n = -1) \qquad \int u^{-1}\ du = \int \frac{du}{u} = \ln|u|$$

$$\int e^u\ du = e^u$$

$$\int \cos(u)\ du = \sin(u)$$

$$\int \sin(u)\ du = -\cos(u)$$

In all cases a constant of integration must be added if the integral is not used to evaluate a definite integral (one with explicit limits).

Definite Integral Rule

Given:

$$\frac{dF(x)}{dx} = f(x)$$

or

$$dF = f(x)\ dx$$

then

$$F(x)|_a^b = F(b) - F(a) = \int_a^b dF = \int_a^b f(x)\ dx$$

Integration by Parts

It's differentiation of a product, both ways:

$$d(uv) = v\ du + u\ dv$$

Move one term to the other side, rearrange, and integrate both sides to get:

$$\int u\ dv = \int d(uv) - \int v\ du = uv - \int v\ du$$

The Chain Rule

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

u-Substitution (Examples)

An application of the chain rule. We wish to do $\int e^{-\alpha t}\ dt$. Let $u = -\alpha t$. Thus $du = -\alpha\ dt$. We convert the integral into a u-form by muliplying and dividing by $\frac{du}{dt} = -\alpha$ to turn dt into du:

$$\int e^{-\alpha t}\ dt = \left(\frac{1}{-\alpha}\right) \int e^{-\alpha t}(-\alpha\ dt) = -\frac{1}{\alpha}e^{-\alpha t}$$

Similarly:

$$\int cos(\omega t)\ dt = \frac{1}{\omega}\sin(\omega t)$$

$$\int (3x+2)^2\ dx = \left(\frac{1}{3}\right)\frac{(3x+2)^3}{3} = \frac{1}{9}(3x+2)^3$$

$$\int \frac{dv}{v - mg/b} = \ln|v - mg/b|$$

Taylor Series

Works best for "small" x:

$$f(u) = f(a+x) = f(a) + \left.\frac{df}{dx}\right|_a x + \left.\frac{d^2 f}{dx^2}\right|_a \frac{x^2}{2!} + ...$$

Binomial Expansion

A special case of the Taylor Series for $f(u) = u^n = (1+x)^n$, expanded for $u = 1 + x$ around 1. Requires $|x| < 1$ to unconditionally converge:

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + ...$$

One Sheet Algebra Review

Solving Simple Equations for an Unknown

For single linear equations, perform operations to both sides of an equation to isolate the unknown. Find x:

$$mx + b = c \quad \rightarrow \quad mx = c - b \quad \rightarrow \quad x = (c-b)/m$$

For simple quadratic equations, you can sometimes ***factor*** it into two linear equations and solve each linear equation. Find the two x roots:

$$ax^2 + bx = 0 \quad \rightarrow \quad (ax+b)x = 0$$

Each factor can equal zero independently, so:

$$x = 0, -b/a$$

are the two roots. Or find the two t roots:

$$3t^2 + 2t - 1 = 0 \quad \rightarrow \quad (3t-1)(t+1) = 0$$

Each factor can equal zero independently, so:

$$t = 1/3, -1$$

These tricks only work by inspection for simple quadratics. The ***quadratic formula***, derived from ***completing the square***, will ***always*** work. Given:

$$ax^2 + bx + c = 0$$

the two roots are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For the second example above, $a = 3$, $b = 2$, $c = -1$ and:

$$t_\pm = \frac{-2 \pm \sqrt{4+12}}{6} = (1/3)_{(+)}, (-1)_{(-)}$$

Best Practice for Radicals

Do not leave radicals in the denominator of a fraction. ***Rationalize*** the expression by multiplying by a clever form of 1:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Here are two simple examples of multiply or factoring, moving things into or out of radicals:

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$$

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{8}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$$

Powers and Exponentials

Here are some very useful things to remember about powers. Suppose a, b, m, and n are ordinary nonzero numbers. Then:

$$a^n \cdot a^m = a^{m+n} \qquad a^n \cdot b^n = (ab)^n$$

a or b can be zero most of the time, but beware negative exponents and (for example) $0^{-1} = 1/0$ which is undefined or maybe "infinity"! Another useful rule:

$$(a^n)^m = a^{n \cdot m}$$

There are three "bases" worth knowing: 2 (binary), e (exponential), and 10 (decimal/fingers). The bases are all postive, hence the absolute magnitude signs in the expressions below. Let's consider the exponential function e^x and its inverse, the natural log $\ln(x)$. Two important true facts worth remembering:

$$e = 2.718281828... \approx 2.7 \quad \text{and} \quad \ln(2) = 0.6931472... \approx 0.7$$

Then:

$$f = e^x \quad \rightarrow \quad \ln|f| = \ln|e^x| = x$$
$$g = e^y \quad \rightarrow \quad \ln|g| = \ln|e^x| = y$$
$$fg = e^x e^y = e^{x+y}$$

or we get the very important result concerning logs:

$$\ln|fg| = \ln|e^{x+y}| = x + y = \ln|f| + \ln|g|$$

Also:

$$1/g = g^{-1} = (e^y)^{-1} = e^{-y} \quad \rightarrow \quad \ln|g^{-1}| = -y = -\ln|g|$$
$$\ln|f/g| = \ln|f| - \ln|g|$$

A similar argument can be used to show that:

$$\ln|y^n| = n\ \ln|y|$$

in general. This is an important property of logarithms (in any base) complementary to the exponential product rules above.

Log functions, natural or otherwise, map multiplication into addition! Here's how we might put this to use to analyze ***exponential decay***. Find the *half life* $t_{1/2}$ when $A(t) = A_0 e^{-t/7}$:

$$A_0/2 = A_0 e^{-t/7} \quad \rightarrow \quad 1/2 = e^{-t/7}$$
$$\ln|1/2| = \ln|e^{-t/7}| = -t/7$$
$$t_{1/2} = -7\ln(1/2) = 7\ln(2) \approx 4.9$$

Here's another useful example in base 10:

$$\log_{10}|2| \approx 0.3 \quad \text{so, e.g.} \quad \log_{10}|200| = \log_{10}|2 \cdot 10^2|$$
$$= \log_{10}|2| + \log_{10}|10^2| = 0.3 + 2 = 2.3$$

One Sheet Simultaneous Equation Review

Things to Consider

- You ***must*** have as many **independent** equations as you have **unknowns**. If you have three unknowns and only two equations, keep looking, think about constraint equations or missing physics.
- Your "answer" for one unknown ***may not*** contain other unknowns. If it does, ***the system is not solved!*** This is a ***common mistake!*** Find your answer in terms of the ***given/known quantities only!***
- Check the units of your answers! Let me put that more clearly as it applies to ALL of the algebraic work you do on the basis of this guide:

CHECK THE UNITS OF YOUR ANSWERS!

Simple Substitution or Elimination

Suppose you know (as the result of applying some physical reasoning) that at some given time t_f:

$$x_f = \frac{1}{2}at_f^2 + v_0 t_f + x_0$$

and

$$v_f = at_f + v_0$$

where x_0, x_f, a and v_0 all are given, but v_f and t_f are not known. We would like to find v_f. To find it, we have to *eliminate* the unknown t_f between the two equations. We could rearrange the first equation (which has only one unknown in it) and solve for t_f in terms of the givens, and substitute it into the second, but that involves the quadratic formula. It is simpler to solve the second equation for t_f in terms of v_f and givens, and substitute this into the first equation:

$$t_f = (v_f - v_0)/a$$

$$(x_f - x_0) = \frac{1}{2}a(v_f - v_0)^2/a^2 + v_0(v_f - v_0)/a$$

$$v_f^2 - v_0^2 = 2a(x_f - x_0)$$

(where you should fill in the missing steps of algebra). Other times it is even simpler:

$$a = (m_1 - m_2\sin(\theta))g/(m_1 + m_2) \qquad \text{and}$$

$$\alpha = a/R \quad \text{so} \quad \alpha = (m_1 - m_2\sin(\theta))g/(m_1 + m_2)R$$

Gauss Elimination and Back Substitution

This is the meat and potatoes approach for linear problems. It is the way computers often solve the problem (with a few bells and whistles). It involves lining equations up so that their variables are right above one another. Then it uses the following reasoning: Multiplying a true equation by a constant produces a true equation. Add two true equations produces a (possibly new) true equation. So we multiply equations by scale factors so that adding or subtracting pairs causes terms with the unknowns to disappear, one at a time, until only one is left and the equation can be solved. One then ***back substitutes*** the result into the preceding step (where you had *two* unknowns, now only one) and solve for the *next* unknown, repeating until all unknowns are known!

I'll give a single example, corresponding to a falling mass unrolling a rope coiled around a massive disk to make it spin up. The unknowns are a, α and T (don't worry yet about what these mean). The system is:

$$mg - T = ma$$

$$RT = \beta MR^2\alpha$$

$$\alpha = a/R$$

The knowns are m, M, R, g. Substitute the third equation into the second to eliminate α immediately. The second becomes:

$$T = \beta Ma$$

Now *add* these two equations:

$$mg - T = ma$$

$$+(T = \beta Ma)$$

to cancel T and get:

$$mg = (m + \beta M)a$$

Solve for a:

$$a = mg/(m + \beta M)$$

Back substitute this into the equation for T above:

$$T = \beta Mmg/(m + \beta M)$$

and α:

$$\alpha = mg/(m + \beta M)R$$

Finally, we check units. Hmmm, mass units cancel, g has units of acceleration, T has an extra mass and hence is a force, α is inverse time squared, all correct. We're good to go!

One Sheet Line Integral Review

Motivation

In physics we have a number of occasions to integrate quantities along a ***specific directed path***. Sometimes the quantity of interest is a scalar, such as mass density to find the total mass of e.g. a piece of string. More often it is used to integrate ***forces or fields*** (both vector quantities) along a ***vector*** path. In particular, this occurs when evaluating ***work, potential energy, and potential***, where the latter is the potential energy per unit mass or charge for the gravitational field or electrostatic field. Line integrals of this sort appear in ***Maxwell's Equations***, which describe the fundamental electromagnetic field.

Definitions

Let C be a curved path of finite length in space. Imagine chopping the curve into a large (eventually infinite) number N of pieces of length Δs. Then:

$$\int_C f(x,y,z)ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta s_i$$

is the *integral of* $f(x,y,z)$ *along the curve* C. In order to use our existing skills in one dimensional integration, we usually have to express ds in terms of a *parameter* that plays the role of i in this sum and then integrate over that parameter.

Example: Find the total mass M of a piece of string with uniform mass density λ shaped like a circular arc of radius R from $\theta = 0$ to $\theta = \pi/2$.

Solution: An infinitesimal chunk of the string has length $ds = \sqrt{dx^2 + dy^2}$ and hence mass $dm = \lambda\ ds$. With considerable effort we could express and perform this integral in terms of cartesian coordinates. However, since R is constant, it is much easier to express ds in terms of the single parameter θ:

$$ds = Rd\theta$$

and

$$\begin{aligned} M &= \int dm = \int \lambda ds \\ &= \lambda \int_0^{\pi/2} R\ d\theta \\ &= \frac{\pi R\lambda}{2} \end{aligned}$$

This makes sense, since the length of the circular arc is $\frac{\pi R}{2}$.

Vector Application

Suppose $\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{x}} + F_y\hat{\boldsymbol{y}} + F_z\hat{\boldsymbol{z}}$ where the $F_i(x,y,z)$ are functions of the general space coordinates x, y, z. In order to evaluate $\int_C \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{\ell}}$ along the curve C we once again have to break the curve up into infinitesimal vector chunks of scalar length $d\ell$, each directed tangent to the curve. Let us write:

$$d\vec{\boldsymbol{\ell}} = d\ell\ \hat{\boldsymbol{\ell}} = d\ell_x\hat{\boldsymbol{x}} + d\ell_y\hat{\boldsymbol{y}} + d\ell_z\hat{\boldsymbol{z}}$$

Then

$$\int_C \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{\ell}} = \int_C (F_x d\ell_x + F_y d\ell_y + F_z d\ell_z)$$

In order to make this integral doable using ordinary one-dimensional integration techniques, we usually (again) parameterize the pieces in terms of an independent variable e.g. s. That is, let each point (x,y,z) on the curve C be a function of a one-dimensional monotonic parameter s: $(x(s), y(s), z(s))$. Then we can write the integral in terms of s:

$$\begin{aligned} \int_C \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{\ell}} &= \int_{s_0}^{s_1} F_x(x(s),y(s),z(s))\frac{d\ell_x}{ds}ds \\ &+ F_y(x(s),y(s),z(s))\frac{d\ell_y}{ds}ds \\ &+ F_z(x(s),y(s),z(s))\frac{d\ell_z}{ds}ds \end{aligned}$$

We only evaluate this for simple cases in an introductory class. For example, suppose $\vec{\boldsymbol{B}} = \frac{\mu_0 I}{2\pi r}(+\sin(\theta)\hat{\boldsymbol{x}} - \cos(\theta)\hat{\boldsymbol{y}})$ (clockwise and tangent to a circle of radius r) and the vector curve C is a circle of radius R directed counterclockwise. Then we can parameterize C with θ such that $d\vec{\boldsymbol{\ell}} = Rd\theta(-\sin(\theta)\hat{\boldsymbol{x}} + \cos(\theta)\hat{\boldsymbol{y}})$ and

$$\int_C \vec{\boldsymbol{B}}\cdot d\vec{\boldsymbol{\ell}} = -\frac{\mu_0 I}{2\pi R}\int_0^{2\pi} R(\sin^2(\theta) + \cos^2(\theta))d\theta = -\mu_0 I$$

An even simpler example is the change in gravitational potential energy of a particle of mass m moving from height y_0 to height y_1 along *any* path:

$$\begin{aligned} \Delta U &= -\int_C \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{\ell}} = -\int_C -mg\hat{\boldsymbol{y}}\cdot d\vec{\boldsymbol{\ell}} = \int_{y_0}^{y_1} mg dy \\ &= mg(y_1 - y_0) \end{aligned}$$

One Sheet Area Integral Review

Motivation

In physics we have a number of occasions to integrate quantities ***over a surface***. Sometimes the quantity of interest is a scalar, such as a surface charge density to find the total charge in e.g. a spherical sheet of charge. However, it is also used to integrate the flux (flow) of a vector field *through* a surface. Area integrals of this latter sort appear in ***Maxwell's Equations***, which describe the fundamental electromagnetic field, although they also describe the vector flow of fluids or electrical currents or radiated energy and much more.

Definitions

Let S be a bounded smooth surface in space. Imagine chopping the surface into a large (eventually infinite) number N of small pieces, each with an area ΔA. Then:

$$\int_S f(x,y,z)dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta A_i$$

is the *integral of* $f(x,y,z)$ *over the surface* S. In order to use our existing skills in one dimensional integration, we usually express dA in terms of two one-dimensional continuous *parameters* that together play the role of i in this sum and then integrate over those parameters. Doing this integral analytically is very difficult or impossible in all by the very simplest of cases.

These simple cases are those where the surface in question is *flat* in 2 dimensions and suitably bounded in separable coordinates (rectangles or polar wedges) or is a suitably bounded curved surface of symmetry in three dimensions – spherical or cylindrical surface segments. We will confine our attention to these cases.

Example: Find the total mass M of a planar uniform surface mass density σ bounded by $\theta \in [0, \pi/2]$ and $r \in [R/2, R]$ in polar coordinates.

Solution: An infinitesimal chunk of the mass has differential area $dA = r\ d\theta\ dr$, the *area element* in polar coordinates. The mass of this chunk is $dm = \sigma dA$. Then:

$$\begin{aligned} M &= \int dm = \int \sigma dA \\ &= \sigma \int_0^{\pi/2} \int_{R/2}^{R} r\ dr\ d\theta = \sigma \left(\int_0^{\pi/2} d\theta\right)\left(\int_{R/2}^{R} r\ dr\right) \\ &= \frac{\pi\sigma}{2}\ \frac{r^2}{2}\bigg|_{R/2}^{R} = \frac{3\pi\sigma}{16} \end{aligned}$$

One can similarly integrate over rectangular areas with area element $dA = dx\ dy$ in cartesian coordinates.

Flux Integrals

Flux is the flow of a vector field *through* a surface. It requires the specification of the direction that will be considered "through". We do this using a ***unit vector perpendicular to the area element*** in the desired direction of "positive" flow. The flux is then the sum/integral of ***the component of the vector field through the surface over the surface***:

$$\Phi_E = \int_S \vec{\boldsymbol{E}}(x,y,z)\cdot\hat{\boldsymbol{n}}\ dA = \lim_{n\to\infty}\sum_{i=1}^{n} E_n(x_i,y_i,z_i)\Delta A_i$$

where E_n is the component of $\vec{\boldsymbol{E}}$ perpendicular to the area element dA at each point in the desired direction.

Example 1: Suppose $\vec{\boldsymbol{E}} = E_0(\sqrt{2}/2\hat{\boldsymbol{x}} + \sqrt{2}/2\hat{\boldsymbol{z}}$ (a uniform field tipped at $\pi/4$ in the x direction from the z direction). What is the downward-directed flux through the rectangle bounded by $x \in [0,a]$, $y \in [0,b]$, when $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{z}}$?

$$\begin{aligned} \Phi_E &= \int_S \vec{\boldsymbol{E}}(x,y,z)\cdot\hat{\boldsymbol{n}}\ dA = -E_0\sqrt{2}/2\int_0^a\int_0^b dxdy \\ &= \frac{E_0\sqrt{2}}{2}\left(\int_0^a dx\right)\left(\int_0^b dy\right) = -\frac{E_0\sqrt{2}ab}{2} \end{aligned}$$

Example 2: Suppose $\vec{\boldsymbol{E}} = E_0\hat{\boldsymbol{z}}$, a uniform field in the z-direction. We wish to find the total upward-directed flux out through the hemispherical surface S defined by $r = R$ above the x-y plane. The area element for a spherical surface in spherical polar coordinates is $dA = r^2\ dr\ \sin(\theta)d\theta\ d\phi$, and $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$ so that $\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{z}} = \cos(\theta)$. Then:

$$\begin{aligned} \Phi_E &= \int_S \vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{n}}dA = E_0\int_0^{\pi/2}\int_0^{2\pi} R^2\cos(\theta)\sin(\theta)d\theta d\phi \\ &= E_0R^2\left(\int_0^{\pi/2}\cos(\theta)\sin(\theta)d\theta\right)\left(\int_0^{2\pi}d\phi\right) \\ &= E_0R^2\left(\frac{1}{2}\right)(2\pi) \\ &= E_0\pi R^2 \end{aligned}$$

This result is easily understood as the projective area perpendicular to $\vec{\boldsymbol{E}}$ is the area of the circle at the base of the hemisphere, πR^2.

In most cases of interest at this level we will be able to evaluate flux as a constant times an area $\Phi_E = E\int dA$ or as zero when $\vec{\boldsymbol{E}} \perp \hat{\boldsymbol{n}}dA$ for all or part of a surface.